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Interlacing property for B-splines

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Abstract

We prove that the zeros of the derivatives of any order of a B-spline are increasing functions of its interior knots. We then prove that if the interior knots of two B-splines interlace, then the zeros of their derivatives of any order also interlace. The same results are obtained for Chebyshevian B-splines.

Keywords: Markov interlacing property; Extended complete Chebyshev spaces; B-splines; Divided differences

1. Introduction

In 1892, Vladimir Markov established the following lemma, now known as the Markov interlacing property.

Lemma 1 (*Markov [7]*). *If the zeros of the polynomial* $p := (\bullet - t_1)\cdots(\bullet - t_n)$ *and the zeros of the polynomial* $q := (\bullet - s_1)\cdots(\bullet - s_n)$ *interlace, that is*

$$t_1 \leqslant s_1 \leqslant t_2 \leqslant s_2 \leqslant \cdots \leqslant t_{n-1} \leqslant s_{n-1} \leqslant t_n \leqslant s_n,$$

then the zeros $\tau_1 \leqslant \cdots \leqslant \tau_{n-1}$ *of* p' *and the zeros* $\sigma_1 \leqslant \cdots \leqslant \sigma_{n-1}$ *of* q' *also interlace, that is*

$$\tau_1 \leqslant \sigma_1 \leqslant \tau_2 \leqslant \sigma_2 \leqslant \cdots \leqslant \tau_{n-1} \leqslant \sigma_{n-1}.$$

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Moreover, if $t_1 < \cdots < t_n$ *and if* $t_i < s_i$ *at least once, then the zeros of* p' *and the zeros of* q' *strictly interlace, that is*

$$\tau_1 < \sigma_1 < \tau_2 < \sigma_2 < \cdots < \tau_{n-1} < \sigma_{n-1}.$$

This lemma plays a major role in the original proof of the Markov inequality [7] and in some of its simplifications, e.g. [2,12]. The interlacing property for perfect splines [1], likewise, is essential in the proof of Markov-type inequalities for oscillating perfect splines [4].

Bojanov remarked that the Markov interlacing property for polynomials is equivalent to a certain monotonicity property, namely

Each zero of the derivative of a polynomial $p := (\bullet - x_1)\cdots(\bullet - x_n)$ *is a strictly increasing function of any* x_j *on the domain* $x_1 < \cdots < x_n$.

He proved [1] this equivalence even for generalized polynomials with respect to a Chebyshev system (satisfying certain conditions), and then obtained the Markov interlacing property for generalized polynomials by showing the monotonicity property.

Bojanov's arguments were somehow similar to the ones used by Vidensky when he gave, in 1951, the following general lemma.

Lemma 2 (*Videnskii [13]*). *Let f and g be two continuously differentiable functions such that any non-trivial linear combination of f and g has at most n zeros counting multiplicity. If the zeros* $t_1 < \cdots < t_n$ *of f and the zeros* $s_1 < \cdots < s_n$ *of g interlace, then* $n-1$ *zeros of* f' *and* $n-1$ *zeros of* g' *strictly interlace.*

In this paper, we aim at proving an interlacing property for B-splines. More precisely, we show that if the interior knots of two polynomial B-splines interlace, then the zeros of their derivatives (of any order) also interlace. In Section 2, we show how this can be derived from what we call the monotonicity property, namely

Each zero of $N^{(l)}_{t_0,\ldots,t_{k+1}}$, $1 \leqslant l \leqslant k-1$, *is a strictly increasing function of any interior knot* t_j, $1 \leqslant j \leqslant k$, *on the domain* $t_0 < t_1 < \cdots < t_k < t_{k+1}$.

This property is proved in Section 3. Next, we generalize these statements to Chebyshevian B-splines. To this end, we need various results which are scattered around the literature and are recalled in Sections 4, 6 and 7. Finally, the proof of the monotonicity property for Chebyshevian B-splines is presented in Section 8.

Our interest in this problem arose from a conjecture regarding the B-spline basis condition number formulated by Scherer and Shadrin [11]. For $\underline{t} = (t_0 < t_1 < \cdots < t_k < t_{k+1})$, with $\omega_{\underline{t}}$ representing the monic polynomial of degree k which vanishes at $t_1, \ldots, t_k$, they asked if it was possible to find a function $\Omega_{\underline{t}}$ vanishing k-fold at t_0 and t_{k+1} and such that the sign pattern of $\Omega^{(l)}_{\underline{t}}$ is the same as the sign pattern of $(-1)^l \omega^{(k-l)}_{\underline{t}}$, $0 \leqslant l \leqslant k$. The hope to choose $\Omega_{\underline{t}}$ as a Chebyshevian B-spline with knots $t_0, \ldots, t_{k+1}$ raised the problem of the monotonicity property. Indeed, the zeros of $\Omega^{(l)}_{\underline{t}}$ should coincide with the zeros of $\omega^{(k-l)}_{\underline{t}}$ and thus should increase with any t_j, $1 \leqslant j \leqslant k$.

Let us mention that the technique we use to establish the monotonicity property for Chebyshevian B-splines is different from the one we use to establish it for polynomial B-splines, so that the proof of Section 3 is redundant. We chose to include it nonetheless because, to our taste, it is a nice proof and because of the additional information it provides, namely Lemma 7.

To simplify the discussion, the notation "$\backsim$" will mean "has the sign of". We will also use the notation $[\![m, n]\!] := \{m, m+1, \dots, n\}$ when m and n are integers.

2. Interlacing property for polynomial B-splines

Let us recall that, for $t_0 \leqslant \cdots \leqslant t_{k+1}$, the L_∞-normalized B-spline of degree k at $t_0, \dots, t_{k+1}$ is defined by

$$N_{t_0,\dots,t_{k+1}}(x) := (t_{k+1} - t_0)\,[t_0, \dots, t_{k+1}](\bullet - x)_+^k,$$

where the divided difference $[t_0, \dots, t_{k+1}]f$ of a function f is the coefficient of degree $k+1$ of the polynomial of degree at most $k+1$ agreeing with f at the points $t_0, \dots, t_{k+1}$. It is well known that, for $\underline{t} := (t_0 < \cdots < t_{k+1})$, the B-spline $N_{\underline{t}}$ is a function of class $\mathcal{C}^{k-1}$ which is positive on (t_0, t_{k+1}) and vanishes elsewhere. The derivative $N_{\underline{t}}^{(k)}$ is constant on each interval (t_i, t_{i+1}), where it has the sign $(-1)^i$. Moreover, for $l \in [\![1, k-1]\!]$, the function $N_{\underline{t}}^{(l)}$ has exactly l interior zeros and it changes sign at these zeros.

We intend to prove that these zeros satisfy an interlacing property with respect to the knots, the first and last knots being fixed, with, say, $t_0 = 0$ and $t_{k+1} = 1$. Let us note that a Vidensky-type argument (where zeros would be allowed to coalesce) is not applicable in this case. Indeed, for two knot sequences $\underline{t}$ and $\underline{t}'$, there is a linear combination of $f := N_{\underline{t}}$ and $g := N_{\underline{t}'}$, namely $\frac{1}{\|f\|} f - \frac{1}{\|g\|} g$, which has more zeros than f does.

Our approach consists of deducing the interlacing property from the monotonicity property. The latter is formulated as follow.

Theorem 1. *For $l \in [\![1, k-1]\!]$, let $0 < s_1 < \cdots < s_l < 1$ be the l interior zeros of $N_{t_0,\dots,t_{k+1}}^{(l)}$. For each $i \in [\![1, l]\!]$, we have*

$$\frac{\partial s_i}{\partial t_j} > 0, \qquad j \in [\![1, k]\!].$$

We note that each s_i is indeed a differentiable function of any t_j. This is derived, using the implicit function theorem, from the fact that $N_{t_0,\dots,t_{k+1}}^{(l+1)}(s_i) \neq 0$. The proof of Theorem 1 is the object of Section 3. If we assume this result for the moment, we can prove the interlacing property for polynomial B-splines.

Theorem 2. *Let $l \in [\![1, k-1]\!]$. If the knots $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$ interlace with the knots $0 = t'_0 < t'_1 < \cdots < t'_k < t'_{k+1} = 1$, that is*

$$t_1 \leqslant t'_1 \leqslant t_2 \leqslant t'_2 \leqslant \cdots \leqslant t_k \leqslant t'_k$$

and if $t_i < t'_i$ *at least once, then the interior zeros* $s_1 < \cdots < s_l$ *of* $N^{(l)}_{t_0,\ldots,t_{k+1}}$ *strictly interlace with the interior zeros* $s'_1 < \cdots < s'_l$ *of* $N^{(l)}_{t'_0,\ldots,t'_{k+1}}$, *that is*

$$s_1 < s'_1 < s_2 < s'_2 < \cdots < s_l < s'_l.$$

Proof. We proceed by induction on l.

For $l = 1$, we just have to show that $s < s'$, where s is the zero of $N'_{\underline{t}}$ and s' is the zero of $N'_{\underline{t}'}$, the knot sequences $\underline{t}$ and $\underline{t}'$ satisfying the interlacing conditions. This follows from Theorem 1.

Let us now assume that the result holds up to an integer $l-1$, $l \in [\![2, k-1]\!]$, and let us prove that it holds for l as well.

Let the knot sequences $\underline{t}$ and $\underline{t}'$ satisfy the interlacing conditions, and let $s_1 < \cdots < s_l$ and $s'_1 < \cdots < s'_l$ denote the interior zeros of $N^{(l)}_{\underline{t}}$ and $N^{(l)}_{\underline{t}'}$, respectively. Theorem 1 yields $s_i < s'_i$ for all $i \in [\![1, l]\!]$. It remains to show that $s'_i < s_{i+1}$ for all $i \in [\![1, l-1]\!]$. To this end, let us assume that $s_{h+1} \leqslant s'_h$ for some $h \in [\![1, l-1]\!]$ and let us derive a contradiction.

First of all, let us remark that it is enough to consider the case of equality $s_{h+1} = s'_h$. Indeed, if $s_{h+1} < s'_h$, we set $\underline{t}(\lambda) = (1-\lambda)\underline{t} + \lambda\underline{t}'$, $\lambda \in [0, 1]$, so that $\underline{t}(0) = \underline{t}$ and $\underline{t}(1) = \underline{t}'$. We also denote the interior zeros of $N^{(l)}_{\underline{t}(\lambda)}$ by $s_1(\lambda) < \cdots < s_l(\lambda)$. By Theorem 1, the point $s_h(\lambda)$ runs monotonically continuously through the interval $[s_h, s'_h]$ when λ runs through $[0, 1]$. As $s_{h+1} \in (s_h, s'_h)$, there exists $\lambda \in (0, 1)$ for which $s_h(\lambda) = s_{h+1}$. But then $\underline{t}$ and $\underline{t}(\lambda)$ satisfy the interlacing conditions and $s_{h+1} = s_h(\lambda)$. This is the case of equality. We are now going to show that it leads to a contradiction.

Let us indeed suppose that $s_{h+1} = s'_h$. We set $s := s_{h+1} = s'_h$ and we let $0 = z_0 < z_1 < \cdots < z_{l-1} < z_l = 1$ and $0 = z'_0 < z'_1 < \cdots < z'_{l-1} < z'_l = 1$ denote the zeros of $N^{(l-1)}_{\underline{t}}$ and $N^{(l-1)}_{\underline{t}'}$, respectively. We know that $s_i < z_i < s_{i+1}$ and that $s'_i < z'_i < s'_{i+1}$ for all $i \in [\![1, l-1]\!]$. Therefore we have

$$z_1 < \cdots < z_h < s < z'_h < \cdots < z'_{l-1}.$$

We also note that, since $s \in (z_h, z_{h+1})$ and $s \in (z'_{h-1}, z'_h)$, one has

$$N^{(l-1)}_{\underline{t}}(s) \backsim (-1)^h \quad \text{and} \quad N^{(l-1)}_{\underline{t}'}(s) \backsim (-1)^{h-1}.$$

Thus we can introduce the function

$$H := N^{(l-1)}_{\underline{t}} + c\, N^{(l-1)}_{\underline{t}'}, \quad \text{where} \quad c := -\frac{N^{(l-1)}_{\underline{t}}(s)}{N^{(l-1)}_{\underline{t}'}(s)} > 0.$$

By the induction hypothesis, one has $z_i \in (z'_{i-1}, z'_i)$, so that $H(z_i) = c\, N^{(l-1)}_{\underline{t}'}(z_i)$ changes sign for $i \in [\![1, h]\!]$. This gives rise to $h-1$ zeros of H in (z_1, z_h). Likewise, one has $z'_i \in (z_i, z_{i+1})$, so that $H(z'_i) = N^{(l-1)}_{\underline{t}}(z'_i)$ changes sign for $i \in [\![h, l-1]\!]$. This gives rise to $l-h-1$ zeros of H in (z'_h, z'_{l-1}). Counting the double zero of H at s, the function H has at least l interior zeros. Applying Rolle's theorem $k-l+1$ times, we deduce that $\widetilde{H} := H^{(k-l+1)}$ has at least $k+1$ sign changes.

But $\widetilde{H} = N_{\underline{t}}^{(k)} + c\, N_{\underline{t}'}^{(k)}$ is a piecewise constant function. On $[t_i', t_{i+1}]$, it has the sign $(-1)^i$, and on $[t_{i+1}', t_{i+2}]$, it has the sign $(-1)^{i+1}$, so that the intermediate value of $\widetilde{H}$ on $[t_{i+1}, t_{i+1}']$ does not contribute to the number of sign changes of $\widetilde{H}$. Only the values of $\widetilde{H}$ on the intervals $[t_0', t_1], \ldots, [t_k', t_{k+1}]$ have a contribution. Hence $\widetilde{H}$ has exactly k sign changes. This is a contradiction.

We conclude that $s_i' < s_{i+1}$ for all $i \in [\![1, l-1]\!]$, so that the result holds for l.
The inductive proof is now complete. □

3. Monotonicity property for polynomial B-splines

Our proof of the monotonicity property for polynomial B-splines makes an extensive use of an elegant formula which was given by Meinardus et al. [8, Theorem 5] and which was expressed in a slightly different way by Chakalov [5] as early as 1938 (see also [3, Formula (3.4.6)]). For the convenience of the reader, we include a proof which, unlike [8], does not involve the integral representation of divided differences.

Lemma 3. *Let* $t_0 = 0$, $t_{k+1} = 1$, *and let* $t \in [0, 1]$, *e.g.* $t_j \leqslant t \leqslant t_{j+1}$. *We have*

$$N_{t_0,\ldots,t_j,t,t_{j+1},\ldots,t_{k+1}}(x) = \frac{x-t}{k+1} N'_{t_0,\ldots,t_j,t,t_{j+1},\ldots,t_{k+1}}(x) + N_{t_0,\ldots,t_{k+1}}(x). \tag{1}$$

Proof. Let us write $\underline{t} := (t_0, \ldots, t_{k+1})$ and $\underline{t}' := (t_0, \ldots, t_j, t, t_{j+1}, \ldots, t_{k+1})$. We define polynomials p, q and r by the facts that

$$\begin{aligned} &p, \text{ of degree } \leqslant k+1, \text{ interpolates } (\bullet - x)_+^k \quad \text{at } \underline{t},\\ &q, \text{ of degree } \leqslant k+2, \text{ interpolates } (\bullet - x)_+^{k+1} \text{ at } \underline{t}',\\ &r, \text{ of degree } \leqslant k+2, \text{ interpolates } (\bullet - x)_+^k \quad \text{at } \underline{t}'. \end{aligned}$$

In this way, since $t_0 = 0$ and $t_{k+1} = 1$,

$$\begin{aligned} &\text{the coefficient of degree } k+1 \text{ of } p \text{ is } N_{\underline{t}}(x),\\ &\text{the coefficient of degree } k+2 \text{ of } q \text{ is } N_{\underline{t}'}(x),\\ &\text{the coefficient of degree } k+2 \text{ of } r \text{ is } -\frac{1}{k+1} N'_{\underline{t}'}(x). \end{aligned}$$

We observe that

$$(\bullet - x) \times r, \text{ of degree } \leqslant k+3, \text{ interpolates } (\bullet - x)_+^{k+1} \text{ at } \underline{t}'. \tag{2}$$

We also remark that the polynomial $r - p$ is of degree at most $k+2$ and vanishes at $\underline{t}$ and that the polynomial $(\bullet - x) \times r - q$ is of degree at most $k+3$ and vanishes at $\underline{t}'$. Looking at the leading coefficients of these polynomials, we obtain

$$\begin{aligned} r - p &= -\frac{1}{k+1} N'_{\underline{t}'}(x) \times (\bullet - t_0) \cdots (\bullet - t_{k+1}),\\ (\bullet - x) \times r - q &= -\frac{1}{k+1} N'_{\underline{t}'}(x) \times (\bullet - t)(\bullet - t_0) \cdots (\bullet - t_{k+1}). \end{aligned}$$

Eliminating r from these equations, we get

$$q - (\bullet - x) \times p = -\frac{1}{k+1} N'_{\underline{t}'}(x) \times (t - x) \times (\bullet - t_0) \cdots (\bullet - t_{k+1}).$$

Identifying the terms of degree $k+2$ leads to

$$N_{\underline{t}'}(x) - N_{\underline{t}}(x) = \frac{x - t}{k+1} N'_{\underline{t}'}(x),$$

which is just a rearrangement of (1). □

Remark 4. The trivial observation (2) is specific to the polynomial case. We will later see how it can be used to simplify the arguments presented in the proof of the monotonicity property for Chebyshevian B-splines.

The two following formulae are crucial in our approach.

Formulae 5. *Using the notations*

$$\underline{t} := (0 = t_0 < \cdots < t_{k+1} = 1), \ \underline{t}^j := (0 = t_0 < \cdots < t_j = t_j < \cdots < t_{k+1} = 1),$$

we have

$$\frac{k+1-l}{k+1} N^{(l)}_{\underline{t}^j}(x) = \frac{x - t_j}{k+1} N^{(l+1)}_{\underline{t}^j}(x) + N^{(l)}_{\underline{t}}(x), \quad l \in [\![0, k-1]\!], \tag{3}$$

and

$$\frac{\partial N^{(m)}_{\underline{t}}}{\partial t_j} = -\frac{1}{k+1} N^{(m+1)}_{\underline{t}^j}. \tag{4}$$

Proof. We rewrite (1) for $t = t_j$ to obtain

$$N_{\underline{t}^j}(x) = \frac{x - t_j}{k+1} N'_{\underline{t}^j}(x) + N_{\underline{t}}(x).$$

Differentiating the latter l times, we obtain formula (3). Formula (4) is an easy consequence of the identity $\frac{\partial}{\partial t_j}[t_0, \ldots, t_{k+1}] = [t_0, \ldots, t_j, t_j, \ldots, t_{k+1}]$. □

To give a feeling of the arguments involved in the proof of the monotonicity property, we begin with the simple case of the zero of the first derivative of a B-spline.

Proposition 6. *Let s be the interior zero of $N'_{t_0,\ldots,t_{k+1}}$, $k \geqslant 2$. We have*

$$\frac{\partial s}{\partial t_j} > 0, \quad j \in [\![1, k]\!].$$

Proof. Differentiating $N'_{\underline{t}}(s) = 0$ with respect to t_j, we get

$$\frac{\partial s}{\partial t_j} \times N''_{\underline{t}}(s) + \left(\frac{\partial N'_{\underline{t}}}{\partial t_j} \right)(s) = 0.$$

Since $N''_{\underline{t}}(s) < 0$, it is enough to show that $\left(\frac{\partial N'_{\underline{t}}}{\partial t_j} \right)(s) > 0$, or, in view of (4), that

$$N''_{\underline{t}^j}(s) < 0.$$

Writing (3) for $l = 1$ and $x = s$, we obtain

$$\frac{k}{k+1} N'_{\underline{t}^j}(s) = \frac{s - t_j}{k+1} N''_{\underline{t}^j}(s). \tag{5}$$

Besides, (3) taken for $l = 0$ and $x = s$ gives

$$\frac{s - t_j}{k+1} N'_{\underline{t}^j}(s) = N_{\underline{t}^j}(s) - N_{\underline{t}}(s).$$

Hence,

$$\frac{(s - t_j)^2}{k+1} N''_{\underline{t}^j}(s) = k \left[N_{\underline{t}^j}(s) - N_{\underline{t}}(s) \right].$$

Let σ be the interior zero of $N'_{\underline{t}^j}$, i.e. the point of maximum of $N_{\underline{t}^j}$. We clearly have $N''_{\underline{t}^j}(\sigma) < 0$. Thus, if $s = \sigma$, we obtain the desired inequality $N''_{\underline{t}^j}(s) < 0$. We can therefore assume that $s \neq \sigma$.

In this case, we can also assume that $s \neq t_j$. Indeed, if $s = t_j$, then (5) would give $N'_{\underline{t}^j}(s) = 0$, so that $s = \sigma$.

Consequently, in order to prove that $N''_{\underline{t}^j}(s) < 0$, we just have to prove that $\left[N_{\underline{t}^j}(s) - N_{\underline{t}}(s) \right] < 0$.

From (3) for $l = 0$ and $x = \sigma$, one has $N_{\underline{t}^j}(\sigma) = N_{\underline{t}}(\sigma)$, and then

$$N_{\underline{t}^j}(s) < N_{\underline{t}^j}(\sigma) = N_{\underline{t}}(\sigma) \leqslant N_{\underline{t}}(s),$$

hence the inequality $\left[N_{\underline{t}^j}(s) - N_{\underline{t}}(s) \right] < 0$ holds. □

A little more work is required in order to adapt these arguments to the case of higher derivatives. The following lemma is needed.

Lemma 7. *Let* $l \in [\![2, k-1]\!]$ *and let* $0 < z_1 < \cdots < z_{l-1} < 1$ *be the zeros of* $N_{\underline{t}}^{(l-1)}$.
(1) *Let* $0 < \sigma_1 < \cdots < \sigma_l < 1$ *denote the zeros of* $N_{\underline{t}^j}^{(l)}$, *we have*

$$\sigma_1 < z_1 < \sigma_2 < z_2 < \cdots < \sigma_{l-1} < z_{l-1} < \sigma_l.$$

(2) *Let* $0 < \zeta_1 < \cdots < \zeta_{l-1} < 1$ *denote the zeros of* $N_{\underline{t}^j}^{(l-1)}$ *and let* $r \in [\![0, l-1]\!]$ *be such that* $\zeta_r < t_j < \zeta_{r+1}$ *(having set* $\zeta_0 := 0$ *and* $\zeta_l := 1$*), we have*

$$z_1 < \zeta_1 < z_2 < \zeta_2 < \cdots < z_r < \zeta_r < \zeta_{r+1} < z_{r+1} < \zeta_{r+2} < z_{r+2} < \cdots < z_{l-2} < \zeta_{l-1} < z_{l-1}.$$

In other words, repeating the knot t_j *moves the zeros of the derivatives of the B-spline towards* t_j.

Let us note that the second statement has already been obtained in the particular case $l = 2$ [8, Theorem 6].

Proof. For the first statement, it is enough to show that there is a zero of $N_{\underline{t}}^{(l-1)}$ in each interval (σ_i, σ_{i+1}), $i \in [\![1, l-1]\!]$. To this end, we note that (3) for $l-1$ and $x = \sigma_i$ gives

$$\frac{k+2-l}{k+1} N_{\underline{t}^j}^{(l-1)}(\sigma_i) = N_{\underline{t}}^{(l-1)}(\sigma_i).$$

Since $N_{\underline{t}^j}^{(l-1)}(\sigma_i) \backsim (-1)^{i+1}$, we have $N_{\underline{t}}^{(l-1)}(\sigma_i) \backsim (-1)^{i+1}$, and the result now follows from the intermediate value theorem.

As for the second statement, we note that (3) for $l-1$ and $x = \zeta_i$ gives

$$N_{\underline{t}}^{(l-1)}(\zeta_i) = \frac{t_j - \zeta_i}{k+1} N_{\underline{t}^j}^{(l)}(\zeta_i).$$

Since $N_{\underline{t}^j}^{(l)}(\zeta_i) \backsim (-1)^i$, there is at least one zero of $N_{\underline{t}}^{(l-1)}$ in each of the intervals $(\zeta_1, \zeta_2), \ldots, (\zeta_{r-1}, \zeta_r), (\zeta_{r+1}, \zeta_{r+2}), \ldots, (\zeta_{l-2}, \zeta_{l-1})$. The result is now clear for $r = 0$ and $r = l-1$. Then, for $r \in [\![1, l-2]\!]$, we have $t_j > \zeta_1$, so that $N_{\underline{t}}^{(l-1)}(\zeta_1) = \frac{t_j - \zeta_1}{k+1} N_{\underline{t}^j}^{(l)}(\zeta_1) < 0$. Besides, we have $N_{\underline{t}}^{(l-1)}(\sigma_1) = \frac{k+2-l}{k+1} N_{\underline{t}^j}^{(l-1)}(\sigma_1) > 0$. Thus there is a zero of $N_{\underline{t}}^{(l-1)}$ in (σ_1, ζ_1). Likewise, there is a zero of $N_{\underline{t}}^{(l-1)}$ in (ζ_{l-1}, σ_l). The $l-1$ zeros of $N_{\underline{t}}^{(l-1)}$ which we have found and localized are simply $z_1, \ldots, z_l$. $\square$

It is now time for the main result of this section.

Theorem 1. *For* $l \in [\![1, k-1]\!]$, *let* $0 < s_1 < \cdots < s_l < 1$ *be the* l *interior zeros of* $N_{t_0,\ldots,t_{k+1}}^{(l)}$. *For each* $i \in [\![1, l]\!]$, *we have*

$$\frac{\partial s_i}{\partial t_j} > 0, \quad j \in [\![1, k]\!].$$

Proof. As the case $l = 1$ has already been treated, we suppose that $l \in [\![2, k-1]\!]$. Differentiating $N_{\underline{t}}^{(l)}(s_i) = 0$ with respect to t_j, we obtain

$$\frac{\partial s_i}{\partial t_j} \times N_{\underline{t}}^{(l+1)}(s_i) + \left(\frac{\partial N_{\underline{t}}^{(l)}}{\partial t_j} \right)(s_i) = 0.$$

Since $N_{\underline{t}}^{(l+1)}(s_i) \backsim (-1)^i$, it is enough to show that $\left(\frac{\partial N_{\underline{t}}^{(l)}}{\partial t_j}\right)(s_i) \backsim (-1)^{i+1}$, or, in view of (4), that

$$N_{\underline{t}^j}^{(l+1)}(s_i) \backsim (-1)^i.$$

Writing (3) for l and $x = s_i$ and for $l-1$ and $x = s_i$, we obtain

$$(k+1-l)N_{\underline{t}^j}^{(l)}(s_i) = (s_i - t_j)N_{\underline{t}^j}^{(l+1)}(s_i),$$
$$(s_i - t_j)N_{\underline{t}^j}^{(l)}(s_i) = (k+2-l)N_{\underline{t}^j}^{(l-1)}(s_i) - (k+1)N_{\underline{t}}^{(l-1)}(s_i).$$

Thus,

$$(s_i - t_j)^2 N_{\underline{t}^j}^{(l+1)}(s_i) = (k+1-l)\left[(k+2-l)N_{\underline{t}^j}^{(l-1)}(s_i) - (k+1)N_{\underline{t}}^{(l-1)}(s_i)\right].$$

Let us suppose that $s_i = t_j$. It is then clear that $l \neq k-1$, and we can write (3) for $l+1$ and $x = s_i$ to obtain $\frac{k-l}{k+1}N_{\underline{t}^j}^{(l+1)}(s_i) = N_{\underline{t}}^{(l+1)}(s_i)$. As $N_{\underline{t}}^{(l+1)}(s_i) \backsim (-1)^i$, we have the desired result $N_{\underline{t}^j}^{(l+1)}(s_i) \backsim (-1)^i$. We can therefore assume that $s_i \neq t_j$.

In this case, we can also assume that $s_i \neq \sigma_i$. Indeed, if $s_i = \sigma_i$, then $N_{\underline{t}^j}^{(l)}(s_i) = 0$, and (3) for l and $x = s_i$ would give $(s_i - t_j)N_{\underline{t}^j}^{(l+1)}(s_i) = 0$, where $N_{\underline{t}^j}^{(l+1)}(s_i) = N_{\underline{t}^j}^{(l+1)}(\sigma_i) \neq 0$, so we would have $s_i = t_j$.

As $s_i \neq t_j$, in order to prove that $N_{\underline{t}^j}^{(l+1)}(s_i) \backsim (-1)^i$, we just have to prove that $\left[(k+2-l)N_{\underline{t}^j}^{(l-1)}(s_i) - (k+1)N_{\underline{t}}^{(l-1)}(s_i)\right] \backsim (-1)^i$.

Since $N_{\underline{t}}^{(l-1)}(s_i) \backsim (-1)^{i+1}$, the result is clear if $N_{\underline{t}^j}^{(l-1)}(s_i) \backsim (-1)^i$. Hence we assume that $N_{\underline{t}^j}^{(l-1)}(s_i) \backsim (-1)^{i+1}$. This implies that $s_i \in [\zeta_{i-1}, \zeta_i]$. Indeed, if for example $s_i < \zeta_{i-1}$, then $s_i < \zeta_{i-2}$, because $N_{\underline{t}^j}^{(l-1)} \backsim (-1)^i$ on $(\zeta_{i-2}, \zeta_{i-1})$, and Lemma 7 yields $s_i < z_{i-1}$, which is absurd.

Now, noting that (3) for $l-1$ and $x = \sigma_i$ implies $(k+2-l)N_{\underline{t}^j}^{(l-1)}(\sigma_i) = (k+1)N_{\underline{t}}^{(l-1)}(\sigma_i)$, we get

$$\begin{aligned}(k+2-l)|N_{\underline{t}^j}^{(l-1)}(s_i)| &< (k+2-l)\|N_{\underline{t}^j}^{(l-1)}\|_{[\infty,\zeta_{i-1},\zeta_i]} \\ &= (k+2-l)|N_{\underline{t}^j}^{(l-1)}(\sigma_i)| = (k+1)|N_{\underline{t}}^{(l-1)}(\sigma_i)| \\ &\leqslant (k+1)\|N_{\underline{t}}^{(l-1)}\|_{[\infty,z_{i-1},z_i]} = (k+1)|N_{\underline{t}}^{(l-1)}(s_i)|.\end{aligned}$$

Therefore $\left[(k+2-l)N_{\underline{t}^j}^{(l-1)}(s_i) - (k+1)N_{\underline{t}}^{(l-1)}(s_i)\right] \backsim -N_{\underline{t}}^{(l-1)}(s_i) \backsim (-1)^i$. □

4. A reminder on ECT-spaces

To formulate the subsequent results, we have to recall a few facts about extended complete Chebyshev spaces and to fix the notations. This is the purpose of this section. Its content is all very standard, and the reader is referred to [10], for example, should more details be needed.

An $(n+1)$-dimensional subspace G of $\mathcal{C}^n(I)$, I interval, is said to be an extended Chebyshev space (ET-space) if any non-zero function in G has no more than n zeros counting multiplicity. The space G is an ET-space if and only if it admits a basis $(g_0, \ldots, g_n)$ which is an extended Chebyshev system (ET-system), that is, for any points $t_0 \leqslant \cdots \leqslant t_n$ in I,

$$\overline{D}\begin{pmatrix} g_0 & \cdots & g_n \\ t_0 & \ldots & t_n \end{pmatrix} := \begin{vmatrix} g_0^{(d_0)}(t_0) & \ldots & g_n^{(d_0)}(t_0) \\ \vdots & \ldots & \vdots \\ g_0^{(d_n)}(t_n) & \ldots & g_n^{(d_n)}(t_n) \end{vmatrix} > 0, \tag{6}$$

the occurrence sequence $\underline{d}$ of $\underline{t}$ being defined by $d_i := \max\{j : t_{i-j} = \cdots = t_i\}$.

The system $(g_0, \ldots, g_n)$ of elements of $\mathcal{C}^n(I)$ is said to be an extended complete Chebyshev system (ECT-system) if $(g_0, \ldots, g_m)$ is an ET-system for any $m \in [\![1, n]\!]$, and an $(n+1)$-dimensional subspace G of $\mathcal{C}^n(I)$ is said to be an extended complete Chebyshev space (ECT-space) if it admits a basis $(g_0, \ldots, g_n)$ which is an ECT-system.

If $(g_0, \ldots, g_n)$ is an ECT-system, given points $t_1 \leqslant \cdots \leqslant t_n$ in I, there exists a unique $\omega \in \operatorname{span}(g_0, \ldots, g_n)$ whose coordinate on g_n is 1 and which satisfies

$$\omega^{(d_i)}(t_i) = 0, \qquad d_i = \max\{j : t_{i-j} = \cdots = t_i\}, \quad i \in [\![1, n]\!].$$

It is denoted $\omega^{g_0,\ldots,g_n}(\bullet; t_1, \ldots, t_n)$, and is given by

$$\omega^{g_0,\ldots,g_n}(\bullet; t_1, \ldots, t_n) = (-1)^n \frac{\begin{vmatrix} g_0 & \cdots & g_{n-1} & g_n \\ g_0^{(d_1)}(t_1) & \ldots & g_{n-1}^{(d_1)}(t_1) & g_n^{(d_1)}(t_1) \\ \vdots & \ldots & \vdots & \vdots \\ g_0^{(d_n)}(t_n) & \ldots & g_{n-1}^{(d_n)}(t_n) & g_n^{(d_n)}(t_n) \end{vmatrix}}{\begin{vmatrix} g_0^{(d_1)}(t_1) & \ldots & g_{n-1}^{(d_1)}(t_1) \\ \vdots & \ldots & \vdots \\ g_0^{(d_n)}(t_n) & \ldots & g_{n-1}^{(d_n)}(t_n) \end{vmatrix}}. \tag{7}$$

According to (6), we easily read the sign pattern of $\omega^{g_0,\ldots,g_n}(\bullet; t_1, \ldots, t_n)$.

Given weight functions $w_0, \ldots, w_n$ such that $w_i \in \mathcal{C}^{n-i}(I)$ and $w_i > 0$ and given a point $t \in I$, we now introduce generalized powers, following the notations used by Lyche [6]. We start by defining inductively the functions $\mathcal{I}_m(\bullet, t) = \mathcal{I}_m(\bullet, t, w_1, \ldots, w_m)$, $m \in [\![0, n]\!]$, by

$$\mathcal{I}_0(\bullet, t) := 1,$$
$$\mathcal{I}_m(\bullet, t, w_1, \ldots, w_m) := \int_t^\bullet w_1(x)\mathcal{I}_{m-1}(x, t, w_2, \ldots, w_m)\, dx.$$

Using integration by parts, it is easily shown by induction that

$$\mathcal{I}_m(x, t) = (-1)^m \mathcal{I}_m(t, x). \tag{8}$$

We then set $u_m(\bullet, t, w_0, \ldots, w_m) := w_0(\bullet)\mathcal{I}_m(\bullet, t, w_1, \ldots, w_m)$, that is

$$\begin{aligned} u_0(x, t, w_0) &= w_0(x), \\ u_1(x, t, w_0, w_1) &= w_0(x) \int_t^x w_1(x_1)\, dx_1, \\ &\vdots \\ u_n(x, t, w_0, \ldots, w_n) &= w_0(x) \int_t^x w_1(dx_1) \cdots \int_t^{x_{n-1}} w_n(x_n)\, dx_n \ldots dx_1. \end{aligned}$$

For example, $u_m(x, t, 1, 1, 2, \ldots, m) = (x - t)^m$.

The system $(u_0(\bullet, t, w_0), \ldots, u_n(\bullet, t, w_0, \ldots, w_n))$ is an ECT-system, and we write $\mathrm{ECT}(w_0, \ldots, w_n)$ for the space it spans, as it indeed is independent on t. In fact, any $(n + 1)$-dimensional ECT-space admits such a representation. In this context, the successive differentiations are to be replaced by the more appropriate ones,

$$\begin{aligned} L_{w_0} &= D\left(\frac{\bullet}{w_0}\right), & D_{w_1,w_0} &= \frac{1}{w_1} L_{w_0}, \\ &\vdots \\ L_{w_{n-1},\ldots,w_0} &= D\left(\frac{\bullet}{w_{n-1}}\right) \circ \cdots \circ D\left(\frac{\bullet}{w_0}\right), & D_{w_n,\ldots,w_0} &= \frac{1}{w_n} L_{w_{n-1},\ldots,w_0}, \end{aligned}$$

so that $L_{w_0}(\mathrm{ECT}(w_0, \ldots, w_n))$ is an ECT-space, namely it is $\mathrm{ECT}(w_1, \ldots, w_n)$.

5. Monotonicity property in ECT-spaces

The Markov interlacing property in ECT-spaces is not new, see e.g. [1]. Here is yet another proof of it, or rather, of the monotonicity property. It is particularly suited to ECT-spaces and we present it for the sole reason that we like it.

Let $\mathrm{ECT}(w_0, \ldots, w_n)$ be an ECT-space on I, and let us set $(u_0, \ldots, u_n) := (u_0(\bullet, t, w_0), \ldots, u_n(\bullet, t, w_0, \ldots, w_n))$ for some $t \in I$. Given $t_1 < \cdots < t_n$ in I, let ω stand here for

$$\omega_{w_0,\ldots,w_n}(\bullet; t_1, \ldots, t_n) := \omega^{u_0,\ldots,u_n}(\bullet; t_1, \ldots, t_n).$$

We define τ_i to be the zero of $L_{w_0}(\omega) \in \mathrm{ECT}(w_1, \ldots, w_n)$ which belongs to the interval (t_i, t_{i+1}), $i \in [\![1, n-1]\!]$.

Proposition 8. *For each* $i \in [\![1, n-1]\!]$, *we have*

$$\frac{\partial \tau_i}{\partial t_j} > 0, \quad j \in [\![1, n]\!].$$

Proof. Dividing by w_0, we can without loss of generality replace w_0 by 1 and L_{w_0} by the usual differentiation. We note that ω is proportional to

$$f := \begin{vmatrix} u_0 & \dots & u_n \\ u_0(t_1) & \dots & u_n(t_1) \\ \vdots & \dots & \vdots \\ u_0(t_n) & \dots & u_n(t_n) \end{vmatrix}, \quad \text{thus we have}$$

$$f'(\tau_i) = \begin{vmatrix} u_0'(\tau_i) & \dots & u_n'(\tau_i) \\ u_0(t_1) & \dots & u_n(t_1) \\ \vdots & \dots & \vdots \\ u_0(t_n) & \dots & u_n(t_n) \end{vmatrix} = 0. \tag{9}$$

Differentiating $f'(\tau_i) = 0$ with respect to t_j leads to

$$\frac{\partial \tau_i}{\partial t_j} \times f''(\tau_i) + \left(\frac{\partial f'}{\partial t_j} \right)(\tau_i) = 0.$$

Note that $f(\tau_i) \backsim (-1)^i$, so that $f''(\tau_i) \backsim (-1)^{i+1}$, hence it is enough to show that

$$\left(\frac{\partial f'}{\partial t_j} \right)(\tau_i) = \begin{vmatrix} u_0'(\tau_i) & u_1'(\tau_i) & \dots & u_n'(\tau_i) \\ \vdots & \dots & \dots & \vdots \\ u_0(t_{j-1}) & u_1(t_{j-1}) & \dots & u_n(t_{j-1}) \\ u_0'(t_j) & u_1'(t_j) & \dots & u_n'(t_j) \\ u_0(t_{j+1}) & u_1(t_{j+1}) & \dots & u_n(t_{j+1}) \\ \vdots & \dots & \dots & \vdots \end{vmatrix} \backsim (-1)^i.$$

Let us introduce

$$g := \begin{vmatrix} u_0'(\tau_i) & u_1'(\tau_i) & \dots & u_n'(\tau_i) \\ \vdots & \dots & \dots & \vdots \\ u_0(t_{j-1}) & u_1(t_{j-1}) & \dots & u_n(t_{j-1}) \\ u_0 & u_1 & \dots & u_n \\ u_0(t_{j+1}) & u_1(t_{j+1}) & \dots & u_n(t_{j+1}) \\ \vdots & \dots & \dots & \vdots \end{vmatrix} \in \mathrm{ECT}(w_0, \dots, w_n),$$

so that $\left(\frac{\partial f'}{\partial t_j} \right)(\tau_i) = g'(t_j)$. We have $g(t_1) = 0, \dots, g(t_{j-1}) = 0, g(t_{j+1}) = 0, \dots, g(t_n) = 0$, and in addition $g(t_j) = 0$, in view of (9). Therefore $g = c\,f$ for some constant c. Using the fact that $(u_0, \dots, u_n)$ is an ECT-system, interchanging the rows yield $g(\tau_i) \backsim (-1)^j$. Now, since $f(\tau_i) \backsim (-1)^i$, we obtain $c \backsim (-1)^{i+j}$. Hence, we get $g'(t_j) = c\,f'(t_j) \backsim (-1)^{i+j}(-1)^j = (-1)^i$, which concludes the proof. □

6. Generalized divided differences

Before turning our attention to the zeros of the derivatives of Chebyshevian B-splines, we need to define these Chebyshevian B-splines. A prerequisite is the introduction of the generalized divided differences, which is carried out in this section. Proposition 11 is of particular importance for our purpose and seems to be new.

For an $(n+1)$-dimensional ECT-space on I, for points $t_0 \leqslant \cdots \leqslant t_n$ in I and for a differentiable enough function f, we write $P^{G}_{t_0,\ldots,t_n}(f)$ for the interpolator of f at $t_0,\ldots,t_n$ in G, i.e. the unique element of G agreeing with f at $t_0,\ldots,t_n$.

Definition 9. The divided difference of a function f at the points $t_0,\ldots,t_n$ with respect to the weight functions $w_0,\ldots,w_n$ is defined (independently on $t \in I$) by

$$\begin{bmatrix} w_0 & \ldots & w_n \\ t_0 & \ldots & t_n \end{bmatrix} f := \text{ coordinate of } P^{\mathrm{ECT}(w_0,\ldots,w_n)}_{t_0,\ldots,t_n}(f) \text{ on } u_n(\bullet, t, w_0,\ldots,w_n).$$

Given $\underline{t} = (t_0 < \cdots < t_n)$, let $\underline{t}_{\backslash i}$ represent the sequence $\underline{t}$ from which t_i has been removed, and $\underline{t}_{\backslash i,j}$ the sequence $\underline{t}$ from which t_i and t_j have been removed. The identity

$$P^{\mathrm{ECT}(w_0,\ldots,w_n)}_{\underline{t}}(f) = P^{\mathrm{ECT}(w_0,\ldots,w_{n-1})}_{\underline{t}_{\backslash i}}(f) + \begin{bmatrix} w_0 & \ldots & w_n \\ t_0 & \ldots & t_n \end{bmatrix} f \times \omega_{w_0,\ldots,w_n}(\bullet; \underline{t}_{\backslash i})$$

is readily obtained, and provides the recurrence relation for divided differences. This can be found in [9], expressed a little differently.

Proposition 10. *For $t_0 < \cdots < t_n$ in I and $0 \leqslant i < j \leqslant n$, we define $\alpha = \alpha^{w_0,\ldots,w_n}_{t_0,\ldots,t_n}(i,j)$ by*

$$\omega_{w_0,\ldots,w_n}(\bullet; \underline{t}_{\backslash j}) - \omega_{w_0,\ldots,w_n}(\bullet; \underline{t}_{\backslash i}) = \alpha \times \omega_{w_0,\ldots,w_{n-1}}(\bullet; \underline{t}_{\backslash i,j}).$$

The constant α is positive and we have

$$\begin{bmatrix} w_0 & \ldots & w_n \\ t_0 & \ldots & t_n \end{bmatrix} = \frac{\begin{bmatrix} w_0 & \ldots & \ldots & \ldots & \ldots & w_{n-1} \\ t_0 & \ldots & t_{i-1} & t_{i+1} & \ldots & t_n \end{bmatrix} - \begin{bmatrix} w_0 & \ldots & \ldots & \ldots & \ldots & w_{n-1} \\ t_0 & \ldots & t_{j-1} & t_{j+1} & \ldots & t_n \end{bmatrix}}{\alpha}.$$

Proof. To simplify the notations, we omit to write the weight functions w_m. The previous identity used twice gives

$$\begin{aligned} P_{\underline{t}}(f) &= P_{\underline{t}_{\backslash i,j}}(f) + \left[\underline{t}_{\backslash j}\right] f \times \omega(\bullet; \underline{t}_{\backslash i,j}) + [\underline{t}] f \times \omega(\bullet; \underline{t}_{\backslash j}) \\ &= P_{\underline{t}_{\backslash i,j}}(f) + \left[\underline{t}_{\backslash i}\right] f \times \omega(\bullet; \underline{t}_{\backslash i,j}) + [\underline{t}] f \times \omega(\bullet; \underline{t}_{\backslash i}). \end{aligned}$$

By subtraction, we get

$$\begin{aligned}\left([\underline{t}_{\backslash i}]f - [\underline{t}_{\backslash j}]f\right) \times \omega(\bullet;\underline{t}_{\backslash i,j}) &= [\underline{t}]f \times \left(\omega(\bullet;\underline{t}_{\backslash j}) - \omega(\bullet;\underline{t}_{\backslash i})\right)\\ &= [\underline{t}]f \times \alpha\,\omega(\bullet;\underline{t}_{\backslash i,j}),\end{aligned}$$

so that $[\underline{t}]f = \frac{1}{\alpha}\left([\underline{t}_{\backslash i}]f - [\underline{t}_{\backslash j}]f\right)$, which is the required result.
The positiveness of α is obtained by taking the values at t_j of the functions defining α. □

In the traditional polynomial case, with $w_0 = 1, w_1 = 1, \ldots, w_n = n$, one easily finds $\alpha^{1,1,2,\ldots,n}_{t_0,\ldots,t_n}(i,j) = t_j - t_i$. Hence we recover the usual definition of divided differences.

Finally, the following result is crucial in our further considerations.

Proposition 11. *For $t_0 < \cdots < t_n$ in I, we have, for any $j \in [\![0,n]\!]$,*

$$\frac{\partial}{\partial t_j}\left(\begin{bmatrix} w_0 & \ldots & w_n \\ t_0 & \ldots & t_n \end{bmatrix}\right) = \beta \begin{bmatrix} w_0 & \ldots & \ldots & \ldots & \ldots & w_{n+1} \\ t_0 & \ldots & t_j & t_j & \ldots & t_n \end{bmatrix},$$

where

$$\beta = \beta^{w_0,\ldots,w_{n+1}}_{t_0,\ldots,t_n}(j) := \frac{\omega'_{w_0,\ldots,w_{n+1}}(t_j;\underline{t})}{\omega_{w_0,\ldots,w_n}(t_j;\underline{t}_{\backslash j})} > 0.$$

Proof. We omit to write the weight functions w_m here as well. For ε small enough, with $\underline{t} := (t_0,\ldots,t_n)$ and $\underline{t}(\varepsilon) := (t_0,\ldots,t_j+\varepsilon,\ldots,t_n)$, we have

$$\begin{aligned}&\frac{1}{\varepsilon}\left(\begin{bmatrix} t_0 & \ldots & t_j+\varepsilon & \ldots & t_n \end{bmatrix} - \begin{bmatrix} t_0 & \ldots & t_j & \ldots & t_n \end{bmatrix}\right)\\ &\quad = \frac{\alpha}{\varepsilon}\begin{bmatrix} t_0 & \ldots & t_j & t_j+\varepsilon & \ldots & t_n \end{bmatrix},\end{aligned}$$

where $\omega(\bullet;\underline{t}) - \omega(\bullet;\underline{t}(\varepsilon)) = \alpha \times \omega(\bullet;\underline{t}_{\backslash j})$. Thus,

$$\frac{\alpha}{\varepsilon} = \frac{-\omega(t_j;\underline{t}(\varepsilon))}{\varepsilon\,\omega(t_j;\underline{t}_{\backslash j})} \underset{\varepsilon\to 0}{\longrightarrow} \frac{\omega'(t_j;\underline{t})}{\omega(t_j;\underline{t}_{\backslash j})}.$$

The latter limit is obtained using (7). The conclusion now follows from the fact that the generalized divided difference depends continuously on the knots, which was shown by Mühlbach [9]. □

For the polynomial case, a simple calculation gives $\beta^{1,1,2,\ldots,n+1}_{t_0,\ldots,t_n}(j) = 1$.

7. Chebyshevian B-splines

We recall some properties of Chebyshevian B-splines that are to play a major role in the last section. We make the simplifying assumption that each w_m is of class $\mathcal{C}^\infty$.

The divided difference (note the reversed order of the w_m's) $\begin{bmatrix} w_{n+1} & \ldots & w_0 \\ t_0 & \ldots & t_{n+1} \end{bmatrix} f$ depends only on $D_{w_0,\ldots,w_{n+1}}(f)$. Indeed, if $D_{w_0,\ldots,w_{n+1}}(f) = D_{w_0,\ldots,w_{n+1}}(g)$, then

$f - g \in \mathrm{ECT}(w_{n+1}, \dots, w_1)$, so that $\begin{bmatrix} w_{n+1} & \dots & w_0 \\ t_0 & \dots & t_{n+1} \end{bmatrix}(f - g) = 0$. Therefore we can define a continuous linear functional λ on $\mathcal{C}([t_0, t_{n+1}])$ such that

$$\begin{bmatrix} w_{n+1} & \dots & w_0 \\ t_0 & \dots & t_{n+1} \end{bmatrix} f = \lambda\left(D_{w_0,\dots,w_{n+1}}(f)\right).$$

One can prove that $\begin{bmatrix} w_{n+1} & \dots & w_0 \\ t_0 & \dots & t_{n+1} \end{bmatrix} f = D_{w_0,\dots,w_{n+1}}(f)(t)$ for some $t \in [t_0, t_{n+1}]$, implying that λ is a positive linear functional of norm 1 on $\mathcal{C}([t_0, t_{n+1}])$. We then expect the representation

$$\begin{bmatrix} w_{n+1} & \dots & w_0 \\ t_0 & \dots & t_{n+1} \end{bmatrix} f = \int_{t_0}^{t_{n+1}} D_{w_0,\dots,w_{n+1}}(f)(t)\, M^{w_0,\dots,w_n}_{t_0,\dots,t_{n+1}}(t)\, dt$$
$$\text{for some } M^{w_0,\dots,w_n}_{t_0,\dots,t_{n+1}} \geqslant 0 \text{ satisfying } \int_{t_0}^{t_{n+1}} M^{w_0,\dots,w_n}_{t_0,\dots,t_{n+1}}(t)\, dt = 1.$$

This is the Peano representation of the divided difference. The choice

$$f = u_n^+(\bullet, x, w_{n+1}, \dots, w_1) := \begin{cases} 0 & \text{on } (-\infty, x) \\ u_n(\bullet, x, w_{n+1}, \dots, w_1) & \text{on } [x, +\infty) \end{cases}$$

for which $L_{w_1,\dots,w_{n+1}}(f) = \delta_x$, leads to

$$M^{w_0,\dots,w_n}_{t_0,\dots,t_{n+1}}(x) = w_0(x) \times \begin{bmatrix} w_{n+1} & \dots & w_0 \\ t_0 & \dots & t_{n+1} \end{bmatrix} u_n^+(\bullet, x, w_{n+1}, \dots, w_1).$$

This identity is taken to be the definition of the Chebyshevian B-spline at $t_0, \dots, t_{n+1}$ with respect to $w_0, \dots, w_n$, and the Peano representation can now be derived from there. The B-spline $M^{w_0,\dots,w_n}_{t_0,\dots,t_{n+1}}$ is positive on (t_0, t_{n+1}) and vanishes elsewhere. For $t_0 < \dots < t_{n+1}$, it is of class $\mathcal{C}^{n-1}$ and its pieces on each interval (t_i, t_{i+1}) are (restrictions of) elements of $\mathrm{ECT}(w_0, \dots, w_n)$. This explains the reversed order of the w_m's and the use of the differentiation L_{w_0}.

One easily get, with the help of (8),

$$L_{w_0}\left(M^{w_0,\dots,w_n}_{t_0,\dots,t_{n+1}}\right)(x)$$
$$= -w_1(x) \times \begin{bmatrix} w_{n+1} & \dots & w_0 \\ t_0 & \dots & t_{n+1} \end{bmatrix} u_{n-1}^+(\bullet, x, w_{n+1}, \dots, w_2). \tag{10}$$

Then the recurrence relation for divided differences implies the differentiation formula

$$L_{w_0}\left(N^{w_0,\dots,w_n}_{t_0,\dots,t_{n+1}}\right)(x) = M^{w_1,\dots,w_n}_{t_0,\dots,t_n}(x) - M^{w_1,\dots,w_n}_{t_1,\dots,t_{n+1}}(x),$$
$$\text{where } N^{w_0,\dots,w_n}_{t_0,\dots,t_{n+1}} := \alpha^{w_{n+1},\dots,w_0}_{t_0,\dots,t_{n+1}}(0, n+1) \times M^{w_0,\dots,w_n}_{t_0,\dots,t_{n+1}}.$$

Applications of the differentiation formula ($l + 1$ times) and of Descartes' rule of sign on the one hand, and application of Rolle's theorem on the other, yields the fact that $L_{w_l,\dots,w_0}\left(M^{w_0,\dots,w_n}_{t_0,\dots,t_{n+1}}\right)$ possesses exactly $l + 1$ interior zeros, where it changes sign.

Remark 12. Some particular attention should be devoted to the interesting case of the B-spline $M^{w}_{t_0,\dots,t_{n+1}} := M^{1,\dots,1,w}_{t_0,\dots,t_{n+1}}$, which is a function of class $\mathcal{C}^{n-1}$, positive on (t_0, t_{n+1}), vanishing elsewhere, and such that its nth (usual) derivative is, on each interval (t_i, t_{i+1}), a multiple of w with sign $(-1)^i$. Given $\underline{t} = (t_0 < \cdots < t_{n+1})$, with $\omega_{\underline{t}}$ denoting the monic polynomial of degree n vanishing at $t_1, \ldots, t_n$, it would be interesting to know if one can choose w so that the l zeros of $(M_{\underline{t}}^w)^{(l)}$ coincide with the l zeros of $\omega_{\underline{t}}^{(n-l)}$. This would confirm the conjecture of Scherer and Shadrin mentioned in the introduction, and would in turn provide the bound for the B-spline basis condition number conjectured by de Boor. Let us note that the cases of small values of n reveal that w cannot be chosen independently on $\underline{t}$.

8. Interlacing property for Chebyshevian B-splines

In this section we finally state and prove the monotonicity property and the interlacing property for Chebyshevian B-splines and the zeros of their appropriate derivatives.

Let us first emphasize two formulae which are essential in our approach.

Formulae 13.

$$\begin{aligned} &L_{w_l,\ldots,w_0}\left(M^{w_0,\ldots,w_n}_{t_0,\ldots,t_{n+1}}\right)(x) \\ &\quad = (-1)^{l+1} w_{l+1}(x) \begin{bmatrix} w_{n+1} & \ldots & w_0 \\ t_0 & \ldots & t_{n+1} \end{bmatrix} u^+_{n-l-1}(\bullet, x, w_{n+1}, \ldots, w_{l+2}), \end{aligned} \tag{11}$$

$$\frac{\partial M^{w_0,\ldots,w_n}_{t_0,\ldots,t_{n+1}}}{\partial t_j} = -\beta^{w_{n+1},\ldots,w_{-1}}_{t_0,\ldots,t_{n+1}}(j) \times L_{w_{-1}}\left(M^{w_{-1},w_0,\ldots,w_n}_{t_0,\ldots,t_j,t_j,\ldots,t_{n+1}}\right). \tag{12}$$

Proof. Formula (11) is obtained in the same way as (10). Formula (12) is obtained from Proposition 11. □

Let us now establish a preparatory lemma.

Lemma 14. *For all $n \geqslant 2$, there holds*

Property A_n. *For any $l \in [\![0, n-2]\!]$, if $s_1 < \cdots < s_{l+1}$ denote the zeros of $L_{w_l,\ldots,w_0}\left(M^{w_0,\ldots,w_n}_{\tau_0,\ldots,\tau_{n+1}}\right)$, we have*

$$L_{w_l,\ldots,w_1}\left(M^{w_1,\ldots,w_n}_{\tau_0,\ldots,\tau_n}\right)(s_i) = L_{w_l,\ldots,w_1}\left(M^{w_1,\ldots,w_n}_{\tau_1,\ldots,\tau_{n+1}}\right)(s_i) \backsim (-1)^{i+1}$$

or equivalently

$$\begin{aligned} &\begin{bmatrix} w_{n+1} & \ldots & w_1 \\ \tau_0 & \ldots & \tau_n \end{bmatrix} u^+_{n-l-1}(\bullet, s_i, w_{n+1}, \ldots, w_{l+2}) \\ &\quad = \begin{bmatrix} w_{n+1} & \ldots & w_1 \\ \tau_1 & \ldots & \tau_{n+1} \end{bmatrix} u^+_{n-l-1}(\bullet, s_i, w_{n+1}, \ldots, w_{l+2}) \backsim (-1)^{i+l+1}. \end{aligned}$$

Proof. The equality $L_{w_l,\ldots,w_1}\left(M^{w_1,\ldots,w_n}_{\tau_0,\ldots,\tau_n}\right)(s_i) = L_{w_l,\ldots,w_1}\left(M^{w_1,\ldots,w_n}_{\tau_1,\ldots,\tau_{n+1}}\right)(s_i)$ is simply a consequence of $L_{w_l,\ldots,w_0}\left(M^{w_0,\ldots,w_n}_{\tau_0,\ldots,\tau_{n+1}}\right)(s_i) = 0$. The other equality is now derived with the help of (11). Thus only the sign of $L_{w_l,\ldots,w_1}\left(M^{w_1,\ldots,w_n}_{\tau_0,\ldots,\tau_n}\right)(s_i)$ has to be determined.

Let us note that if A_{n-1} holds, then so does the following property.

Property B_n. For any $m \in [\![1, n-2]\!]$, if f is an element of $\mathrm{ECT}(w_n, \ldots, w_1)$ agreeing with $u_m^+(\bullet, z, w_n, \ldots, w_{n-m})$ at $\tau_0, \ldots, \tau_n$ and if $\ell(f)$ denotes the coordinate of f on $u_{n-1}(\bullet, z, w_n, \ldots, w_1)$, then

$$\left[f - u_m^+(\bullet, z, w_n, \ldots, w_{n-m})\right]_{|(\tau_i, \tau_{i+1})} \backsim \ell(f)(-1)^{n+i}.$$

Indeed, for $m \in [\![1, n-2]\!]$, let us consider such a function f and let us suppose that $\left[f - u_m^+(\bullet, z, w_n, \ldots, w_{n-m})\right]$ has at least $n+2$ zeros counting multiplicity. A repeated application of Rolle's theorem implies that

$$D_{w_{n-m},\ldots,w_n}\left[f - u_m^+(\bullet, z, w_n, \ldots, w_{n-m})\right] = D_{w_{n-m},\ldots,w_n}(f) - u_0^+(\bullet, z)$$

has at least $n+2-m$ zeros, and then, applying Rolle's theorem once more, we see that $L_{w_{n-m},\ldots,w_n}(f)$ vanishes at least $n-m$ times. But $L_{w_{n-m},\ldots,w_n}(f)$ is an element of $\mathrm{ECT}(w_{n-m-1}, \ldots, w_1)$, therefore $L_{w_{n-m},\ldots,w_n}(f) = 0$. The latter implies that $f \in \mathrm{ECT}(w_n, \ldots, w_{n-m}) \subseteq \mathrm{ECT}(w_n, \ldots, w_2)$. Since f agrees with $u_m^+(\bullet, z, w_n, \ldots, w_{n-m})$ at $\tau_1, \ldots, \tau_n$, it follows that

$$\begin{bmatrix} w_n & \ldots & w_1 \\ \tau_1 & \ldots & \tau_n \end{bmatrix} u_m^+(\bullet, z, w_n, \ldots, w_{n-m}) = 0.$$

Consequently, according to A_{n-1}, we have $L_{w_{n-m-2},\ldots,w_0}\left(M_{\tau_0,\ldots,\tau_n}^{w_0,\ldots,w_{n-1}}\right)(z) \neq 0$, i.e.

$$\begin{bmatrix} w_n & \ldots & w_0 \\ \tau_0 & \ldots & \tau_n \end{bmatrix} u_m^+(\bullet, z, w_n, \ldots, w_{n-m}) \neq 0.$$

But this contradicts the fact that $f \in \mathrm{ECT}(w_n, \ldots, w_1)$.
We conclude that $\left[f - u_m^+(\bullet, z, w_n, \ldots, w_{n-m})\right]$ vanishes only at $\tau_0, \ldots, \tau_n$, where it changes sign. For $x \to -\infty$, one has $f(x) - u_m^+(x, z, w_n, \ldots, w_{n-m}) = f(x) \backsim \ell(f)(-1)^{n-1}$, hence the sign pattern given in B_n.

We now proceed with the proof of the lemma.
Firstly, we remark that the assertion in A_n is clear for $l = 0$. Indeed, if s is the zero of $L_{w_0}\left(M_{\tau_0,\ldots,\tau_{n+1}}^{w_0,\ldots,w_n}\right)$, then $s \in (\tau_1, \tau_n)$, and consequently $M_{\tau_0,\ldots,\tau_n}^{w_1,\ldots,w_n}(s) > 0$.

Let us now show A_n by induction on $n \geqslant 2$.

According to the remark we have just made, the assertion A_2 is true.

Let us then suppose that A_{n-1} holds for some $n \geqslant 3$, and let us prove that A_n holds as well.
For $l \in [\![1, n-2]\!]$, let

$$z_0 = \tau_0 < z_1 < \cdots < z_l < z_{l+1} = \tau_n \text{ be the zeros of } L_{w_l,\ldots,w_1}\left(M_{\tau_0,\ldots,\tau_n}^{w_1,\ldots,w_n}\right),$$
$$s_1 < \cdots < s_{l+1} \text{ be the zeros of } L_{w_l,\ldots,w_0}\left(M_{\tau_0,\ldots,\tau_{n+1}}^{w_0,\ldots,w_n}\right).$$

We will have shown A_n as soon as we prove that $s_i \in (z_{i-1}, z_i)$, $i \in [\![1, l+1]\!]$.

We consider

$$\begin{aligned} f_i &\in \mathrm{ECT}(w_{n+1},\dots,w_1) \text{ agreeing with } u^+_{n-l-1}(\bullet,z_i,w_{n+1},\dots,w_{l+2}) \\ &\qquad\qquad \text{at } \tau_1,\dots,\tau_{n+1}, \\ g_i &\in \mathrm{ECT}(w_{n+1},\dots,w_1) \text{ agreeing with } u^+_{n-l-1}(\bullet,z_i,w_{n+1},\dots,w_{l+2}) \\ &\qquad\qquad \text{at } \tau_0,\dots,\tau_n. \end{aligned}$$

Let $\ell(f_i)$ denote the coordinate of f_i on $u_n(\bullet,t,w_{n+1},\dots,w_1)$.
Let also $\ell(g_i)$ denote the coordinate of g_i on $u_{n-1}(\bullet,t,w_{n+1},\dots,w_2)$. In fact, we have $g_i \in \mathrm{ECT}(w_{n+1},\dots,w_2)$, as the coordinate of g_i on $u_n(\bullet,t,w_{n+1},\dots,w_1)$ is equal to zero, in view of $L_{w_l,\dots,w_1}\left(M^{w_1,\dots,w_n}_{\tau_0,\dots,\tau_n}\right)(z_i)=0$. According to A_{n-1}, we have

$$\ell(g_i)=\begin{bmatrix} w_{n+1} & \dots & w_2 \\ \tau_1 & \dots & \tau_n \end{bmatrix} u^+_{n-l-1}(\bullet,z_i,w_{n+1},\dots,w_{l+2}) \backsim (-1)^{i+l}, \quad i\in [\![1,l]\!].$$

Let us now remark that

$$g_i - f_i = -\ell(f_i)\times \omega_{w_{n+1},\dots,w_1}(\bullet;\tau_1,\dots,\tau_n).$$

Therefore,

$$\begin{aligned}(g_i-f_i)(\tau_{n+1}) &= -\ell(f_i)\times\omega_{w_{n+1},\dots,w_1}(\tau_{n+1};\tau_1,\dots,\tau_n)\backsim -\ell(f_i)\\ &= g_i(\tau_{n+1}) - u^+_{n-l-1}(\tau_{n+1},z_i,w_{n+1},\dots,w_{l+2}) \underbrace{\backsim}_{B_n} \ell(g_i).\end{aligned}$$

We conclude that $\ell(f_i)\backsim -\ell(g_i)\backsim(-1)^{i+l+1}$, $i\in[\![1,l]\!]$. In other words,

$$\begin{bmatrix} w_{n+1} & \dots & w_1 \\ \tau_1 & \dots & \tau_{n+1} \end{bmatrix} u^+_{n-l-1}(\bullet,z_i,w_{n+1},\dots,w_{l+2}) \backsim (-1)^{i+l+1},$$

that is

$$L_{w_l,\dots,w_1}\left(M^{w_1,\dots,w_n}_{\tau_1,\dots,\tau_{n+1}}\right)(z_i)\backsim(-1)^{i+1}, \quad i\in[\![1,l]\!].$$

This implies the existence of $r_i\in(z_{i-1},z_i)$ such that

$$L_{w_l,\dots,w_1}\left(M^{w_1,\dots,w_n}_{\tau_0,\dots,\tau_n}\right)(r_i) = L_{w_l,\dots,w_1}\left(M^{w_1,\dots,w_n}_{\tau_1,\dots,\tau_{n+1}}\right)(r_i),$$

i.e.

$$L_{w_l,\dots,w_0}\left(M^{w_0,\dots,w_n}_{\tau_0,\dots,\tau_{n+1}}\right)(r_i)=0, \quad i\in[\![2,l]\!].$$

Let us also note that

$$\begin{aligned} L_{w_l,\dots,w_1}\left(M^{w_1,\dots,w_n}_{\tau_1,\dots,\tau_{n+1}}\right)(z_1) > 0 = L_{w_l,\dots,w_1}\left(M^{w_1,\dots,w_n}_{\tau_0,\dots,\tau_n}\right)(z_1),\\ L_{w_l,\dots,w_1}\left(M^{w_1,\dots,w_n}_{\tau_1,\dots,\tau_{n+1}}\right)(\tau_1) = 0 < L_{w_l,\dots,w_1}\left(M^{w_1,\dots,w_n}_{\tau_0,\dots,\tau_n}\right)(\tau_1),\end{aligned}$$

hence the existence of $r_1\in(\tau_1,z_1)\subseteq(z_0,z_1)$ such that

$$L_{w_l,\dots,w_0}\left(M^{w_0,\dots,w_n}_{\tau_0,\dots,\tau_{n+1}}\right)(r_1)=0.$$

Similarly, we get the existence of $r_{l+1} \in (z_l, z_{l+1})$ such that

$$L_{w_l,\ldots,w_0}\left(M^{w_0,\ldots,w_n}_{\tau_0,\ldots,\tau_{n+1}}\right)(r_{l+1}) = 0.$$

Having found $l+1$ zeros of $L_{w_l,\ldots,w_0}\left(M^{w_0,\ldots,w_n}_{\tau_0,\ldots,\tau_{n+1}}\right)$, these zeros are just $s_1, \ldots, s_{l+1}$. On account of $s_i \in (z_{i-1}, z_i)$, we conclude that A_n holds.
Our inductive proof is now complete. $\square$

Remark 15. The observation made in Remark 4 was central to our proof of the monotonicity property for polynomial B-splines. It also simplifies the arguments we have just given here, as follow.

Proof of Lemma 14 for polynomial B-splines. Let s_i be the ith zero of $M^{(l+1)}_{\tau_0,\ldots,\tau_{n+1}}$. Then the polynomial p interpolating $(\bullet - s_i)^{n-l-1}_+$ at $\tau_0, \ldots, \tau_{n+1}$ is of degree n, not $n+1$. Its leading coefficient is $M^{(l)}_{\tau_1,\ldots,\tau_{n+1}}(s_i)$. As in (2), the polynomial $(\bullet - s_i) \times p$, of degree $n+1$, interpolates $(\bullet - s_i)^{n-l}_+$ at $\tau_0, \ldots, \tau_{n+1}$. Its leading coefficient is $M^{(l)}_{\tau_0,\ldots,\tau_{n+1}}(s_i)$ and is also the leading coefficient of p. Therefore, one has

$$M^{(l)}_{\tau_1,\ldots,\tau_{n+1}}(s_i) = M^{(l)}_{\tau_0,\ldots,\tau_{n+1}}(s_i) \backsim (-1)^{i+1}. \qquad \square$$

We are now ready to establish the monotonicity property for Chebyshevian B-splines.

Theorem 3. *For $l \in [\![0, k-2]\!]$, let $s_1 < \cdots < s_{l+1}$ be the $(l+1)$ interior zeros of $L_{w_l,\ldots,w_0}\left(M^{w_0,\ldots,w_k}_{t_0,\ldots,t_{k+1}}\right)$. For each $i \in [\![1, l+1]\!]$, we have*

$$\frac{\partial s_i}{\partial t_j} > 0, \quad j \in [\![1, k]\!].$$

Proof. Differentiating $D_{w_{l+1},\ldots,w_0}\left(M^{w_0,\ldots,w_k}_{t_0,\ldots,t_{k+1}}\right)(s_i) = 0$ with respect to t_j, we obtain

$$\frac{\partial s_i}{\partial t_j} \times L_{w_{l+1},\ldots,w_0}\left(M^{w_0,\ldots,w_k}_{t_0,\ldots,t_{k+1}}\right)(s_i) + \frac{\partial D_{w_{l+1},\ldots,w_0}\left(M^{w_0,\ldots,w_k}_{t_0,\ldots,t_{k+1}}\right)}{\partial t_j}(s_i) = 0.$$

Since $L_{w_{l+1},\ldots,w_0}\left(M^{w_0,\ldots,w_k}_{t_0,\ldots,t_{k+1}}\right)(s_i) \backsim (-1)^i$, it is enough to show that

$$\frac{\partial D_{w_{l+1},\ldots,w_0}\left(M^{w_0,\ldots,w_k}_{t_0,\ldots,t_{k+1}}\right)}{\partial t_j}(s_i) = D_{w_{l+1},\ldots,w_0}\left(\frac{\partial M^{w_0,\ldots,w_k}_{t_0,\ldots,t_{k+1}}}{\partial t_j}\right)(s_i) \backsim (-1)^{i+1}$$

or, in view of (12) and (11), that

$$\begin{bmatrix} w_{k+1} & \ldots & \ldots & \ldots & w_0 & w_{-1} \\ t_0 & \ldots & t_j & t_j & \ldots & t_{k+1} \end{bmatrix} u^+_{k-l-1}(\bullet, s_i, w_{k+1}, \ldots, w_{l+2}) \backsim (-1)^{i+l}.$$

We consider

$$f \in \mathrm{ECT}(w_{k+1}, \ldots, w_{-1}) \text{ agreeing with } u^+_{k-l-1}(\bullet, s_i, w_{k+1}, \ldots, w_{l+2}) \text{ at } t_0, \ldots, t_j, t_j, \ldots, t_{k+1},$$
$$g \in \mathrm{ECT}(w_{k+1}, \ldots, w_0) \text{ agreeing with } u^+_{k-l-1}(\bullet, s_i, w_{k+1}, \ldots, w_{l+2}) \text{ at } t_0, \ldots, t_{k+1}.$$

Let $\ell(f)$ denote the coordinate of f on $u_{k+2}(\bullet, t, w_{k+1}, \ldots, w_{-1})$. We need to show that

$$\ell(f) \backsim (-1)^{i+l}.$$

Let also $\ell(g)$ denote the coordinate of g on $u_k(\bullet, t, w_{k+1}, \ldots, w_1)$. In fact, we have $g \in \mathrm{ECT}(w_{k+1}, \ldots, w_1)$, which is a consequence of $L_{w_l,\ldots,w_0}\left(M^{w_0,\ldots,w_k}_{t_0,\ldots,t_{k+1}}\right)(s_i) = 0$. According to A_k, we have

$$\ell(g) = \begin{bmatrix} w_{k+1} & \ldots & w_1 \\ t_1 & \ldots & t_{k+1} \end{bmatrix} u^+_{k-l-1}(\bullet, s_i, w_{k+1}, \ldots, w_{l+2}) \backsim (-1)^{i+l+1}.$$

Let us now remark that

$$f - g = \ell(f) \times \omega_{w_{k+1},\ldots,w_{-1}}(\bullet; t_0, \ldots, t_{k+1}).$$

Therefore,

$$f'(t_j) - g'(t_j) = \ell(f) \times \omega'_{w_{k+1},\ldots,w_{-1}}(t_j; t_0, \ldots, t_{k+1}) \backsim \ell(f)(-1)^{k+1+j}$$
$$= -\left(g - u^+_{k-l-1}(\bullet, s_i, w_{k+1}, \ldots, w_{l+2})\right)'(t_j) \underbrace{\backsim}_{B_{k+1}} \ell(g)(-1)^{k+j}.$$

We conclude that $\ell(f) \backsim -\ell(g) \backsim (-1)^{i+l}$. □

The interlacing property for Chebyshevian B-splines is deduced from the monotonicity property in exactly the same way as in Section 2. Its proof is therefore omitted.

Theorem 4. *Let $l \in [\![0, k-2]\!]$. If the knots $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$ interlace with the knots $0 = t'_0 < t'_1 < \cdots < t'_k < t'_{k+1} = 1$, that is*

$$t_1 \leqslant t'_1 \leqslant t_2 \leqslant t'_2 \leqslant \cdots \leqslant t_k \leqslant t'_k,$$

and if $t_i < t'_i$ at least once, then the interior zeros $s_1 < \cdots < s_{l+1}$ of $L_{w_l,\ldots,w_0}\left(M^{w_0,\ldots,w_k}_{t_0,\ldots,t_{k+1}}\right)$ strictly interlace with the interior zeros $s'_1 < \cdots < s'_{l+1}$ of $L_{w_l,\ldots,w_0}\left(M^{w_0,\ldots,w_k}_{t'_0,\ldots,t'_{k+1}}\right)$, that is

$$s_1 < s'_1 < s_2 < s'_2 < \cdots < s_{l+1} < s'_{l+1}.$$

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